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Gravitational instantons, extra dimensions and form fields

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Abstract: A broad class of higher-dimensional instanton solutions are found for a theory which contains gravity, a scalar field and antisymmetric tensor fields of arbitrary rank. The metric used, a warp product of an arbitrary number of any compact Einstein manifolds, includes many of great interest in particle physics and cosmology. For example 4D FRW universes with additional dimensions compactified on a Calabi-Yau three fold, a torus, a compact hyperbolic manifold or a sphere are all included. It is shown that the solution of this form which dominates the Hartle Hawking path integral is always a higher-dimensional generalisation of a Hawking Turok instanton when the potential of the scalar field is such that these instantons can exist. On continuation to lorentzian signature such instantons give rise to a spacetime in which all of the spatial dimensions are of equal size and where the spatial topology is that of a sphere. The extra dimensions are thus not hidden. In the case where the potential for the scalar field is generated solely by a dilatonic coupling to the form fields we find no integrable instantons at all. In particular we find no integrable solutions of the type under consideration for the supergravity theories which are the low energy effective field theories of superstrings.

Keywords: Field Theories in Higher Dimensions, Solitons Monopoles and Instantons, Cosmology of Theories beyond the SM.

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Contents

1. Introduction

Recently a number of papers have appeared in which the instanton cosmology associated with higher-dimensional theories is considered [1]–[8]. Interest in such a union of initial conditions and modern particle physics theories is understandable. Extra hidden dimensions are a vital ingredient of many modern models of particle physics yet there are a number of ways in which a higher-dimensional theory can be dimensionally reduced, leading to many possible effective theories in 4D at low energies. This loss in predictive power could be reversed if a well motivated proposal for the state of the universe existed which picked out a particular compactification scheme.

One such proposal that has received considerable attention is due to Hartle and Hawking [9]. Recently Hawking and Turok [1, 10] obtained a new class of instanton solutions which, using this proposal, could be interpreted as representing the initial configurations for our universe. In an earlier work with Saffin [7] we addressed the issue of the singular instantons arising in higher-dimensional theories, and showed that in the presence of only a scalar field, the most likely configuration we found was always the highly symmetrical Hawking Turok instanton — a possible problem when we wish to have compactified extra spatial dimensions.

In this paper we address the consequences of adding extra matter fields, namely form fields, into the higher-dimensional action. The calculations peformed here and in [7, 11] can be viewed as a necessary complement to the work which addresses the important issue of perturbations about instantons [12]–[18]. Such calculations can show whether or not a given solution is a local minimum of the action. However, there may be other minima of lower action elsewhere in parameter space where the minima under consideration are in different conformal equivalence classes. It would then be one or more of these other minima which would dominate the path integral, not the first solution.

The layout of the paper is as follows. In section 2, we introduce our metric and briefly recap the results for the case where the form fields are absent [7]. In section 3 we proceed to introduce antisymmetric fields through a lagrangian which is a simple field strength squared term. In section 4 we examine the case of more general couplings between the scalar and antisymmetric tensor fields. We also make some comments about the absence of integrable instantons of the type under consideration in the low energy effective field theories of string theory before conluding in section 5.

2. Instantons without form fields

Our starting point is a manifold $\mathcal{M}$ which has a metric structure imposed on it, and a scalar field ϕ living on it with potential $\mathcal{V}(\phi)$. By using the usual torsion free, metric connection on $\mathcal{M}$ we are able to describe the equations of motion for both the metric and ϕ, which follow from the Einstein-Hilbert action.

$$S_{\mathrm{E}} = \int_{\mathcal{M}} \eta \left[-\frac{1}{2\kappa^2}\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + \mathcal{V}(\phi) \right] + \text{boundary term}\,. \tag{2.1}$$

Here, $\kappa^2 = 8\pi/m_{\mathrm{P}l}^2$ (scaled to unity for the rest of the paper), and the boundary terms are such that the action does not contain second derivatives of the metric [19, 20]. η is the volume form and $\mathcal{R}$ is the Ricci scalar of the connection.

We consider the manifold $\mathcal{M}$ as a foliation in euclidean time of a product of boundary-less manifolds. At any given time ξ we can then write $\mathcal{M}(\xi)$ as a Cartesian product of $\mathcal{M}_{(i)}$, $i = 1, \ldots, T$ each with dimension $n_{(i)}$, where for convenience we define $N = n_{(1)} + n_{(2)} + \cdots + n_{(T)}$. To endow $\mathcal{M}$ with a metric structure we start by putting a metric on each of the $\mathcal{M}_{(i)}$, denoted $\mathrm{ds}^2_{(i)}$. The metric structure we impose on $\mathcal{M}$ then follows by introducing a ξ dependent scale factor, $a_{(i)}$, for each $\mathcal{M}_{(i)}$; providing information about the relative size of the $\mathcal{M}_{(i)}$ at any given ξ. The resulting metric is then,

$$\mathrm{ds}^2 = \mathrm{d}\xi^2 + a^2_{(1)}(\xi)\mathrm{ds}^2_{(1)} + a^2_{(2)}(\xi)\mathrm{ds}^2_{(2)} + \cdots + a^2_{(T)}(\xi)\mathrm{ds}^2_{(T)}\,. \tag{2.2}$$

The only restriction on the $ds^2_{(i)}$'s is that they are Einstein metrics on compact manifolds; the Ricci tensor is proportional to the metric, the constant of proportionality being Λ_i for the i-th submanifold. This is necessary if the metric is to be consistent with our ansatz for the scalar field, $\phi = \phi(\xi)$. Of the many solutions that exist, a class of these instantons can be continued to a 4-dimensional inflating universe, with a number of static extra dimensions. This form for the metric is very general. It includes a wide class of metrics commonly considered in cosmology, such as those leading to the Coleman-De Lucia instanton [21], Kantowski-Sachs instantons [22], Hawking-Turok instanton [1], FRW external spaces with compact hyperbolic manifolds as the internal space [23], most of the models of string cosmology arising out of compactifications on tori (for a review see [24]), compactifications of string theory on Calabi-Yau spaces (for a review see [25]), and some compactifications of Supergravity theories on spheres (for a review see [26]). For example in [1, 21] $T = 1$ and $\mathcal{M}_{(1)}$ is a three sphere with its standard round metric. A more exotic Kantowski-Sachs metric was considered in [22], there $T = 2$ with $\mathcal{M}_{(1)} = S^1$, $\mathcal{M}_{(2)} = S^2$.

The equations of motion for this system are as follows (see [7] for details).

$$\frac{1}{2}\left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\right)^2 - \frac{1}{2}\sum_{i=1}^{T} n_i \left(\frac{a_i'}{a_i}\right)^2 - \sum_{i=1}^{T} \frac{n_i \Lambda_i}{2a_i^2} = \frac{1}{2}\phi'^2 - \mathcal{V}\,, \tag{2.3}$$

$$\begin{aligned}\sum_{i=1}^{T} n_i \frac{a_i''}{a_i} - \frac{a_k''}{a_k} + \frac{1}{2}\left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\right)^2 - \sum_{i=1}^{T} n_i \frac{a_i' a_k'}{a_i a_k} -& \\ -\frac{1}{2}\sum_{i=1}^{T} n_i \left(\frac{a_i'}{a_i}\right)^2 + \left(\frac{a_k'}{a_k}\right)^2 - \frac{1}{2}\sum_{i=1}^{T} \frac{n_i \Lambda_i}{a_i^2} + \frac{\Lambda_k}{a_k^2} &= -\frac{1}{2}\phi'^2 - \mathcal{V}\,, \end{aligned} \tag{2.4}$$

$$\phi'' + \sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\phi' = \frac{\partial \mathcal{V}}{\partial \phi}\,, \tag{2.5}$$

where eq. (2.4) is true for all k. By defining $V_{(i)}$ as the volume of $\mathcal{M}_{(i)}$ and β to be the product of the scale factors we find that the action, including boundary terms, is given by

$$S_{\rm E} = V_{(1)} \cdots V_{(T)} \left\{ -\left[\frac{\partial \beta}{\partial \xi}\right]_{\xi_S}^{\xi_N} + \int \mathrm{d}\xi \beta \left[-\frac{1}{2}\mathcal{R} + \frac{1}{2}\phi'^2 + \mathcal{V}(\phi)\right]\right\} \tag{2.6}$$

$$= V_{(1)} \cdots V_{(T)} \left\{ -\left[\frac{\partial \beta}{\partial \xi}\right]_{\xi_S}^{\xi_N} - \frac{2}{n_{(1)} + \cdots + n_{(T)} - 1}\int \mathrm{d}\xi \beta(\xi)\mathcal{V}(\phi)\right\}, \tag{2.7}$$

$$\beta(\xi) = a_{(1)}^{n_{(1)}} a_{(2)}^{n_{(2)}} \cdots a_{(T)}^{n_{(T)}}\,. \tag{2.8}$$

In arriving at the second equation we have used the trace of Einstein's equations to eliminate the scalar curvature and scalar field kinetic terms. The quantities ξ_S and ξ_N refer to the range of the ξ coordinate, with ξ_N being the "north" pole and ξ_S referring to the "south" pole of the instanton taken to be $\xi = 0$. To save writing out the volumes of all the submanifolds we shall call the term in the curly braces of eq. (2.7) the *reduced action*.

Given that the metrics on the $\mathcal{M}_i$'s are Einstein, the evolution equations for the scale factors $a_{(i)}(\xi)$ depend only on the value of the "cosmological constants" $\Lambda_{(i)}$ and not on the detailed topology or geometry of the manifold. This is potentially very significant, for any statements we can make about the evolution of the scale factors cover a very large class of manifolds, all those admitting an Einstein metric.

In [7] it was concluded that the instanton of the above class with minimum euclidean action is a higher-dimensional generalisation of the Hawking Turok solution. It was also conjectured that the action of a configuration could only be negative if all of its submanifolds were positively curved. The reason for this conjecture, which we checked for a wide range of solutions, is that negative or zero curvature submanifolds can have arbitrary volume for a given curvature. Therefore, since the action is proportional to the volume of each of the submanifolds if we had an instanton with negative action and a negatively curved submanifold we could make its action arbitrarily negative by raising the volume of that submanifold. Clearly this would make our instanton approximation to the path integral ill defined. In the case of positively curved sub manifolds this problem does not occur. Bishop's theorem states that for a positively curved Einstein manifold of given curvature the metric which maximises the volume of the manifold is the round sphere. Thus there is an upper limit to the volume of the submanifolds, which is saturated by the Hawking Turok type of solutions.

The conclusion that a Hawking Turok instanton dominates the path integral in these higher-dimensional theories leads to a potential problem. When this instanton is continued to lorentzian signature it gives a spacetime which is a higher-dimensional generalisation of an open FRW. Thus the Hartle Hawking proposal in the case where we have this simple matter content predicts that none of the extra dimensions should be compactified.

While this metric is very general the matter field content, just a minimally coupled scalar field, is not. The obvious thing to study then is whether or not this problem arises in situations where more general forms of matter are present. Preliminary calculations for the cases of multiple and non-minimally coupled scalar fields showed that such simple modifications of the theory considered in [7] are not enough to solve this problem. We therefore turn to another common ingredient of the bosonic sectors of higher-dimensional particle physics theories — antisymmetric tensor fields — will the extra structure they introduce give us more phenomenologically acceptable results?

3. Instantons with form fields

Antisymmetric tensor fields occur commonly in the bosonic sectors of supergravity theories and can be used to obtain solutions where some of the dimensions of the theories are compactified [27]. These fields have also been used in a cosmological setting where the dominant instanton is assumed to be one which gives rise to a compactified spacetime [1, 5, 6].

We shall start therefore by considering the following d-dimensional action (temporarily reinserting κ).

$$S_{\mathrm{E}} = \int_{\mathcal{M}} d^d x \sqrt{g} \left[-\frac{1}{2\kappa^2}\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + \mathcal{V}(\phi) + \frac{1}{2(d_f)!} F_{\mu_1\cdots\mu_{d_f}} F^{\mu_1\cdots\mu_{d_f}} \right] + \\ + \text{GH} + \text{FF boundary terms}\,, \tag{3.1}$$

where the GH (Gibbons Hawking) boundary term is given by,

$$\text{GH boundary term} = \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{h} K \tag{3.2}$$

and the FF (Form Field) boundary term is given by,

$$\text{FF boundary term} = -\int_{\partial\mathcal{M}} d^{d-1} x \sqrt{h} \frac{1}{d_f!} F^{\mu_1\cdots\mu_{d_f}} n_{\mu_1} A_{\mu_2\cdots\mu_{d_f}} \tag{3.3}$$

$A_{\mu_2\cdots\mu_{d_f}}$ here is the antisymmetric potential which gives rise to the field strength F and n is a unit vector normal to the boundary. $K = h^{ij}K_{ij}$ is the trace of the second fundamental form, the trace being taken with the metric, h_{ij}, which has been induced on the boundary by g_{ij}.

The boundary terms are vital. They ensure that the action we are considering leads to the equations of motion of the fields upon variation in a volume when we hold the metric and field strength, but not the derivative of the metric or the gauge potential, constant on the boundary. In other words the boundary terms ensure that the equations of motion in the classical limit follow in the usual way from the path integral [19, 28].

The dimensionality of the antisymmetric tensor field, d_f, is important in determining which subset of the metrics seen in the previous section are consistent with supporting a non-zero field. There are two different Ansatze for the form fields which are consistent with the symmetries of our metric. They are [29],

$$\begin{aligned} F_{0\mu_1\cdots\mu_{d_f-1}} &= \frac{1}{\prod_{i=\text{lit}} a_i^{n_i}} E'(\xi)\epsilon_{\mu_1\cdots\mu_{d_f-1}}\,, \\ F_{\mu_1\cdots\mu_{d_f}} &= \frac{1}{\prod_{i=\text{lit}} a_i^{n_i}} M\epsilon_{\mu_1\cdots\mu_{d_f}}\,. \end{aligned} \tag{3.4}$$

The $i = \text{lit}$ in the products of scalefactors in the denominators simply mean that we only include the scale factors of manifolds which are spanned by the forms.[1] $\epsilon^{\mu_1\cdots\mu_{d_f}}$ is a totally antisymmetric quantity the components of which vanish if any

[1]A submanifold is spanned by a form if the form is non zero when some or all of its indices correspond to coordinates associated with that submanifold.

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of the indices are not on one of the submanifolds spanned by the form and are ± 1 otherwise. M is a constant and E' is a function of ξ. For the sake of simplicity all the results we will present here have only one such ansatz activated per form in any given instanton. The more general case will be the subject of future work.

The first of these two ansatze is the generalisation of an electric field while the second is the generalisation of a magnetic field. We will refer to them as the electric and magnetic cases, respectively from now on.

There are certain rules for combining forms of a given degree with submanifolds of a given dimensionality. These rules come from demanding that our form field ansatz combines with our metric (and scalar field) ansatz in such a way as to be a solution to Einstein's equations. Given the symmetries of our metric we have the following conditions.

- If we have a magnetic ansatz for a form with a p-form field strength then the dimensionalities of a subset of the submanifolds must sum to p.
- If we have an electic ansatz for a form with a p-form field strength then the dimensionalities of a subset of the submanifolds must sum to $p-1$.

These conditions will be added to when we come to consider boundary conditions at the regular pole of our instantons. The magnetic ansatz automatically obeys the form field equations of motion. In the Electric case the equation of motion for E is,

$$E'' + \left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i} - 2 \sum_{i=\mathrm{lit}} n_i \frac{a_i'}{a_i} \right) E' = 0 \,. \tag{3.5}$$

Thus we have,

$$E' = \frac{C}{\prod_{i=1}^{T} a_i^{n_i} \prod_{j=\mathrm{lit}} a_j^{-2n_j}} \,, \tag{3.6}$$

where C is a constant parameter like M. However, unlike M we will take C here to be imaginary. There has been a controversy in the literature over the procedure of taking the electric field to be imaginary on Euclideanisation. There are several reasons for making this choice. Firstly it means that under naive analytical continuation back to lorentzian signature, treating the continuation as an imaginary general coordinate transformation, we will obtain real electric fields rather than imaginary ones. Secondly if this choice is not made (i.e. if the electric field is taken to be real in euclidean space) then the field strength governed by C in the dominant instanton is infinite — the path integral becomes ill defined even when we ignore the conformal factor problem in the gravitational sector. When carrying out the calculations (which are given below for the case of imaginary electric field) to see this it is vital to remember the boundary term for the form field. There are a number of other reasons for this choice of C which are less central to the discussion here. By choosing the electric field to be imaginary we make the processes of dualisation and Wick rotation commute — this is not the case if we take C real. Choosing the electric field to be imaginary seems unavoidable if we want to consider electric/magnetic duality in the euclidean signature theory. For example electrically charged extremal black holes can only exist in the Euclideanised theory if electric charge, and therefore electric field, is imaginary. Magnetically charged extremal black holes however do exist for real magnetic charge. A more detailed discussion of these points can be found in refs. [1, 2, 5, 28, 30, 31, 32].

Writing out the remaining equations of motion explicitly including form field degrees of freedom we obtain,

$$\frac{1}{2}\left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\right)^2 - \frac{1}{2}\sum_{i=1}^{T} n_i \left(\frac{a_i'}{a_i}\right)^2 - \sum_{i=1}^{T}\frac{n_i\Lambda_i}{2a_i^2} = \frac{1}{2}\phi'^2 - \mathcal{V} - \\ - \frac{(M^2 - E'^2)}{2}\frac{1}{\prod_{i=\mathrm{lit}} a_i^{2n_i}}\,, \tag{3.7}$$

$$\sum_{i=1}^{T} n_i \frac{a_i''}{a_i} - \frac{a_k''}{a_k} + \frac{1}{2}\left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\right)^2 - \sum_{i=1}^{T} n_i \frac{a_i' a_k'}{a_i a_k} - \\ -\frac{1}{2}\sum_{i=1}^{T} n_i \left(\frac{a_i'}{a_i}\right)^2 + \left(\frac{a_k'}{a_k}\right)^2 - \frac{1}{2}\sum_{i=1}^{T}\frac{n_i\Lambda_i}{a_i^2} + \frac{\Lambda_k}{a_k^2} = -\frac{1}{2}\phi'^2 - \mathcal{V} \pm \\ \pm \frac{(M^2 + E'^2)}{2},\frac{1}{\prod_{i=\mathrm{lit}} a_i^{2n_i}}\,, \tag{3.8}$$

$$\phi'' + \sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\phi' = \frac{\partial \mathcal{V}}{\partial \phi}\,. \tag{3.9}$$

In eq. (3.8) the upper sign should be used when the submanifold associated with k is lit and the lower sign when it is not. Now using eq. (3.4) we can rewrite the euclidean action of our system as follows.

$$\begin{aligned} S_{\mathrm{E}} &= \text{FF Boundary Term} + \\ &\quad + V_{(1)}\cdots V_{(T)}\left\{-\left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N} + \int \mathrm{d}\xi\,\beta(\xi)\left[-\frac{1}{2}\mathcal{R} + \frac{1}{2}\phi'^2 + \mathcal{V}(\phi)\ + \right.\right. \\ &\quad\left.\left. + \frac{1}{2d_f!}F_{\mu_1\cdots\mu_{d_f}}F^{\mu_1\cdots\mu_{d_f}}\right]\right\} \\ &= \text{FF Boundary Term} + \\ &\quad + V_{(1)}\cdots V_{(T)}\left\{-\left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N} - \frac{1}{n_{(1)} + \cdots + n_{(T)} - 1}\times\right. \\ &\quad\left. \times \int \mathrm{d}\xi\,\beta(\xi)\left(2\mathcal{V}(\phi) - \frac{(d_f - 1)(M^2 + E'^2)}{\prod_{i=lit} a_i^{2n_i}}\right)\right\}, \end{aligned} \tag{3.10}$$

where

$$\beta(\xi) = a_{(1)}^{n_{(1)}} a_{(2)}^{n_{(2)}} \cdots a_{(T)}^{n_{(T)}} . \qquad (3.11)$$

Consider now the form field boundary term.

$$\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\frac{1}{d_f!}F^{\mu_1\cdots\mu_{d_f}} n_{\mu_1} A_{\mu_2\cdots\mu_{d_f}} = \int_{\mathcal{M}} d^d x \partial_{\mu_1}\sqrt{g}\left(\frac{1}{d_f!}F^{\mu_1\cdots\mu_{d_f}} A_{\mu_2\cdots\mu_{d_f}}\right). \qquad (3.12)$$

For the cases we are considering the only "boundary" is in the "0" direction so we may replace this with,

$$\text{FF Boundary term} = -\int_{\mathcal{M}} d^d x \partial_0 \sqrt{g}\left(\frac{1}{d_f!}F^{0\mu_2\cdots\mu_{d_f}} A_{\mu_2\cdots\mu_{d_f}}\right). \qquad (3.13)$$

For an instanton, $\partial_{\mu_1}\sqrt{g}F^{\mu_1\cdots\mu_{d_f}} = 0$ by the form field equations of motion. Thus we have,

$$\text{FF Boundary term} = -\int_{\mathcal{M}} d^d x \sqrt{g}\frac{1}{d_f!}F^{0\mu_2\cdots\mu_{d_f}} \partial_0 A_{\mu_2\cdots\mu_{d_f}} . \qquad (3.14)$$

Now $\partial_0 A_{\mu_2\cdots\mu_{d_f}} = 0$ in the magnetic case so there the boundary term vanishes. In the electric case the boundary term becomes,

$$\text{FF Boundary term} = -V_{(1)}V_{(2)}\cdots V_{(T)}\int_{\mathcal{M}} d^d x \beta(\xi)\frac{1}{d_f!}F_{\mu_1\cdots\mu_{d_f}} F^{\mu_1\cdots\mu_{d_f}} . \qquad (3.15)$$

Thus the boundary term reverses the sign of the electric field term in the bulk action [32]. Finally then we are left with the following expression for the action of our solutions.

$$\begin{aligned} S_E = -\prod_{i=1}^{T} V_{(i)} \Bigg\{ \left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N} &+ \frac{2}{\sum_{i=1}^{T} n_{(i)} - 1} \times \\ &\times \int \mathrm{d}\xi \beta \left(\mathcal{V} - \frac{(d_f-1)(M^2 - E'^2)}{2\prod_{i=lit} a_i^{2n_i}}\right) \Bigg\} . \end{aligned} \qquad (3.16)$$

It should be remembered that E' is imaginary in the above.

If we wish to have one regular pole in our instantons, as opposed to having a singularity at each end, then there are certain rules which the fields must obey at that regular pole. For the metric and scalar field these rules have already been worked out, for the case of higher-dimensional examples, and can be found for example in [7]. Although instantons with two singular poles are possible [33] the interesting features we wish to discuss are found in the case where only the north pole is singular. The rules are as follows.

- The Instanton must be ended by the scale factor of one, and only one, of the submanifolds going to zero — this submanifold "pinches off" the regular pole.

- The submanifold which pinches off the regular pole must be positively curved.
- The scale factor of the submanifold which pinches off the regular pole must approach zero as

$$a_{(1)}(\xi \to 0) \longrightarrow \sqrt{\frac{\Lambda_{(i)}}{n_i - 1}}\ \xi\,, \tag{3.17}$$

as we approach the regular pole.

- All other scale factors must approach a constant at the regular pole.
- The scalar field must approach a constant at the regular pole.

In the case where form fields are present these rules are supplemented by the following derived by demanding regularity and using eqs. (3.6), (3.7) and (3.8).

- In the electric case the form must span the submanifold which pinches off the regular pole. This means that the form can only span one submanifold (and time) and that the submanifold in question must have dimension $p-1$ for a p-form field strength. In particular this means it is impossible to have different dimensional electric field strengths present on the same instanton solution.
- In the magnetic case the form can span any number of submanifolds (subject to the rules already outlined above) as long as none of them pinch off the regular pole.

The equations of motion for the instanton solutions are complicated, so we have solved them numerically. An important constraint is that an instanton solution which makes a sensible contribution to the path integral must have finite action. Thus we check that as we approach the singular pole the fields and scale factors behave in such a way as to render the path "integrable". We did this in [7] for the case of a scalar field in the metric in use here. There it was found that, for a wide class of potentials for the scalar field, the action was always integrable.[2] This is no longer the case once we include the more general form of matter being considered here. For simplicity we consider the case of two submanifolds. Let us assume that near the singularity we have,

$$a_1(\xi \to \xi_N) \propto (\xi - \xi_N)^p\,, \qquad a_2(\xi \to \xi_N) \propto (\xi - \xi_N)^q\,. \tag{3.18}$$

[2]There are examples where this is not the case. For example in the case of higher-dimensional generalisations of the Hawking Turok solutions the analysis of Hawking and Reall [6] can be generalised to give the following result. For the action of an instanton with a potential for its scalar field of the form $V_0 e^{\alpha\phi}$ to be integrable, we require $|\alpha| < 2\sqrt{\frac{d-1}{d-2}}$. The same result for $d = 4$ was also obtained in [34]

We also make the assumption, which we can check is consistent once we have finished, that in eq. (3.7), (3.8) and (3.9) the $\mathcal{V}$ term is negligible near the singularity, integrating eq. (3.9) then gives,

$$\phi'(\xi \to \xi_N) \propto (a_1^{n_1} a_2^{n_2})^{-1} . \tag{3.19}$$

We then split the possible cases into three types, $p > 0$, $p = 0$, and $p < 0$. For $p > 0$ the terms present due to the form fields in the equations of motion and constraint equation are subdominant at the singularity. Consistency of our equations then implies $np+mq = 1$, from which it follows that the action is integrable. For $p = 0$ we again find $np+mq = 1$ but now the form term in the action is found to be infinite. For $p < 0$ we find $mq = 1$ and again the form term in the action is infinite. Thus when we present results below we only plot the action for configurations satisfying $p > 0$. It is amusing to note that while the scalar field diverges at the north pole in the cases where the action is finite it does not in the case $p < 0$ where the action is infinite!

3.1 The effect of form fields on the minimum action instanton

In the numerical work including form fields, for simplicity, we choose a potential $\mathcal{V} = \frac{1}{2}\phi^2$ and specialise to the case of two submanifolds. Our conclusions are not qualitatively dependent on these choices. In ref. [7] we were able to plot the reduced action of our instantons against the 2 parameters which described them, the value of the scalar field and one of the scale factors at the regular pole. Unfortunately such a neat representation of our results is not possible here as the parameter space of solutions for a given dimensionality of form field, d_f, has a maximum dimension of three. The possible parameters are: the value of the non-zero scale factor at the regular pole, the value of the scalar field at the same point and M or C if the dimensionalities of the submanifolds and field strengths are such as to allow the relevant ansatze. To find the solution with minimum reduced action in such a parameter space we adopt the following procedure. The reduced action for each solution is obtained and the set of parameters which give the instanton of lowest reduced action are stored. We then apply a number of checks to make sure that we are obtaining a sensible result. The first of these is that we make sure that the minimum action solution is located well within the volume of parameter space which we have covered. This ensures that we are indeed looking at a minimum rather than just one of the solutions on the edge of our parameter space volume in the direction in which the action is decreasing. Once this is done we can plot a number of 2 parameter cross-sections through the space of solutions at the minimum to make sure everything is well behaved. Figures 1 and 2 are representive cross-sections a little way away from the minimum for the case of two Einstein metrics of dimensions $n_1 = 3$ and $n_2 = 2$ and a 4 form field strength. This means we are allowed an electric

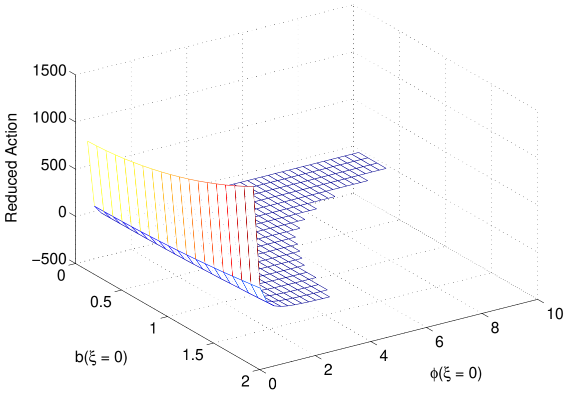

Figure 1: Plot of the reduced action on a cross-section through parameter space corresponding to $C = 0.01i$. In this case $n_{(a)} = 3, n_{(b)} = 2$ and the antisymmetric field strength has four indices.

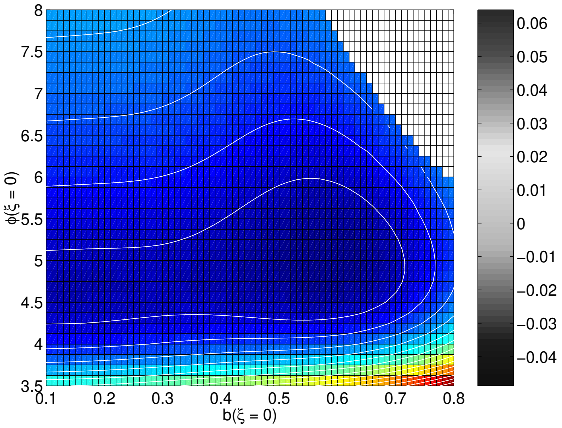

Figure 2: Magnification of a region of figure 1 showing the minimum reduced action configuration on this slice of parameter space.

but not a magnetic field within the instanton. To be specific we have chosen to take the value of $\Lambda_i = n_i - 1$, which is the appropriate value for the round metric on S^{n_i}. We shall explain the reason for this choice and the reason why the cross-sections we have chosen to present are a little way from the minimum shortly.

The second of these plots is a magnification of some structure present on the first, showing a minimum on this slice of parameter space with negative reduced action. The cut out region in the plots corresponds to those configurations which do not have finite action.

Now in order to find the dominant configuration in the Hartle Hawking path integral we should, in principle, obtain the volume factors of the submanifolds in order to calculate the full action. Infact there is no need to do this here. This is because whichever of the metrics considered in this paper we look at, we have found

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that the dominant instanton, that with lowest action, is the one in which C and M — all the form fields — vanish. This is the reason why we didn't plot the cross-sections we presented above so as to go through the minimum. The plots would then have been identical to those in ref. [7]. Given this then we may employ the results of [7] for instantons where the only matter is a scalar field. This leads us to conclude that, even when including a form field, the dominant instanton of the type studied here is once again a higher-dimensional generalisation of the simple Hawking Turok solution where the only matter which does not vanish is a scalar field.

3.2 The extension of a conjecture

In the calculations presented above we have obtained minima of negative reduced action. Thus we can surely obtain instantons of arbitrarily negative action merely by using submanifolds of increasingly large volume? We are saved from this catastrophe by two things. Firstly we use a result from the theory of Einstein manifolds called Bishop's Theorem. It tells us that for a given positive curvature the Einstein metric of largest volume is that of the round sphere. This is why we have plotted results appropriate to round spheres above. Secondly we made a conjecture in [7] that *negative values of the reduced action occur only if* $\Lambda_i > 0$ *for all* i. We now extend this conjecture to the case where forms are present as described in this paper. We have checked this conjecture in numerous cases and have always found it to hold true.

4. Dilatonic couplings to form fields and string effective actions

Instead of the case considered above we could allow for couplings between the dilaton and form fields by considering a lagrangian density for the antisymmetric tensor fields of the form,

$$\mathcal{L}_F F = f(\phi) F_{\mu_1 \cdots \mu_{d_f}} F^{\mu_1 \cdots \mu_{d_f}} \frac{\sqrt{g}}{2 d_f!} , \tag{4.1}$$

where $f(\phi) > 0$, for all ϕ if the Hartle Hawking path integral is to be well defined, even if we ignore the conformal factor problem. This condition is satisfied by the low energy effective actions of string theory. The derivation of the equations of motion which follow from our new action is a straight forward generalisation of the calculation performed in the previous two sections. We obtain the following:

$$E' = \frac{C}{\prod_{i=1}^{T} a_i^{n_i} \prod_{j=\mathrm{lit}} a_j^{-2n_j} f(\phi)} ,$$

$$\frac{1}{2}\left(\sum_{i=1}^{T} n_i \frac{a_i'}{a_i}\right)^2 - \frac{1}{2}\sum_{i=1}^{T} n_i \left(\frac{a_i'}{a_i}\right)^2 - \sum_{i=1}^{T} \frac{n_i \Lambda_i}{2a_i^2} = \frac{1}{2}\phi'^2 - \mathcal{V} -$$

$$
-\frac{(M^2-E'^2)}{2}\frac{f(\phi)}{\prod_{i=\mathrm{lit}} a_i^{2n_i}}\,,
$$

$$
\begin{aligned}
&\sum_{i=1}^{T} n_i\frac{a_i''}{a_i}-\frac{a_k''}{a_k}+\frac{1}{2}\left(\sum_{i=1}^{T} n_i\frac{a_i'}{a_i}\right)^2-\sum_{i=1}^{T} n_i\frac{a_i'a_k'}{a_ia_k}-\\
&-\frac{1}{2}\sum_{i=1}^{T} n_i\left(\frac{a_i'}{a_i}\right)^2+\left(\frac{a_k'}{a_k}\right)^2-\frac{1}{2}\sum_{i=1}^{T}\frac{n_i\Lambda_i}{a_i^2}+\frac{\Lambda_k}{a_k^2}=-\frac{1}{2}\phi'^2-\mathcal{V}\pm\\
&\qquad\qquad\qquad\pm\frac{(M^2+E'^2)}{2}\frac{f(\phi)}{\prod_{i=\mathrm{lit}} a_i^{2n_i}}\,,\\
&\phi''+\sum_{i=1}^{T} n_i\frac{a_i'}{a_i}\phi'=\frac{\partial\mathcal{V}}{\partial\phi}+\frac{(M^2+E'^2)}{2}\times\\
&\qquad\qquad\times\frac{1}{\prod_{i=\mathrm{lit}} a_i^{2n_i}}\frac{\partial f}{\partial\phi}\,,
\end{aligned}
\tag{4.2}
$$

The action may be written as,

$$
S_E=V_{(1)}\cdots V_{(T)}\left\{-\left[\frac{\partial\beta}{\partial\xi}\right]_{\xi_S}^{\xi_N}-\frac{1}{n_{(1)}+\cdots+n_{(T)}-1}\times\right.
$$
$$
\left.\times\int \mathrm{d}\xi\,\beta(\xi)\left(2\mathcal{V}-\frac{(d_f-1)(M^2-E'^2)f(\phi)}{\prod_{i=lit} a_i^{2n_i}}\right)\right\}. \tag{4.3}
$$

It should be remembered that E' is imaginary in the above. As before, we first of all check which of our configurations have finite action. Using eq. (3.18) and assuming $f\propto(\xi-\xi_n)^r$ a similar analysis to that presented earlier shows that for the action to be finite we require $2np>r$. If this condition is fullfilled then $np+mq=1$. ϕ' obeys eq. (3.19) and so assuming that the condition holds and knowing f as a function of ϕ we can calculate r. We can then check for consistency by looking to see that we do indeed have $r<2np$, which means that integrable solutions could exist.

In the case where $\mathcal{V}$ is non zero and gives rise to integrable instantons ($\mathcal{V}=\frac{1}{2}\phi^2$ for example) we once again find that the form fields vanish in the dominant instanton. A more interesting case is that where we have $\mathcal{V}=0$ and the potential for the scalar field is completely generated by its coupling to the antisymmetric tensor fields. The reason for this is twofold. Firstly such a situation arises if we naively try and find instantons which extremise euclidean low energy string effective actions. Secondly it appears at first glance that such a situation should give us dominant instantons with constant ξ cross-sections of non-trivial topology. This is because compact singular instantons can only exist, in the absence of branes, when ϕ has a potential. If ϕ is to have a potential here then the form fields must be non zero. This in turn leads to the instantons necessarily having non-trivial "spatial" topology unless the rank of the form field strength is the same as, or one less than, the dimensionality of spacetime.

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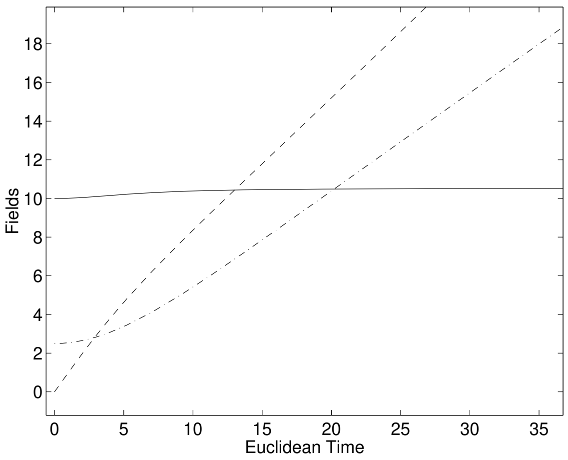

Figure 3: Field profiles for one of the non-compact instantons. The solid line is 10ϕ and the dashed and dot-dashed lines are scale factors of 3- and 2-dimensional spheres, respectively. In this case we have $c = i$ and $f = e^{-0.5\phi}$.

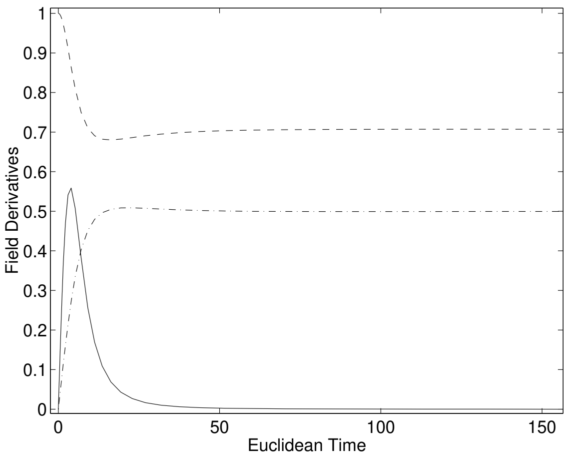

Figure 4: Curvature driven terminal velocities of the fields shown in the previous plot. The solid line is the derivative of the solid line in figure 3 and so on.

Unfortunately once we start to look at the solutions things do not work out quite how we might hope. The problem that arises is that there are no instantons of the type being described here which have finite action.

We numerically integrated our equations as before and checked wide ranges of parameter space looking for integrable instantons, failing to find any for the choices of the function $f(\phi)$ that we tried. For example $f(\phi) = 0.5\phi^2$ or any of the exponential couplings appropriate for a 10-dimensional low energy effective action from string theory. The only types of field profile we obtain are non-integrable ones or configurations like those shown in figures 3 and 4.

The explanation of what is happening is as follows. If the instanton does not close off quickly enough then the scale factor b becoming large "turns off" the potential of the scalar field due to the presence of the form fields. ϕ then skids to a halt and there is no energy to close off the instanton. The two scale factors head for curvature driven terminal velocity and we can see that this is the asymptotic form of the solution. In fact it is easy to predict the late time rate of change of each scale factor. If we put double derivatives and the form and ϕ' terms to zero in our equations and rearrange slightly we obtain the following.

$$(m-1)\left(\frac{b'}{b}\right)^2 + n\frac{a'b'}{ab} - \frac{\Lambda_m}{b^2} = 0\,,$$
$$(n-1)\left(\frac{a'}{a}\right)^2 + m\frac{a'b'}{ab} - \frac{\Lambda_n}{a^2} = 0\,. \tag{4.4}$$

If we now make the assumption $a = kb$ where k is a constant (which will be true at late times if the system approaches terminal velocity) we obtain,

$$a' = \sqrt{\frac{\Lambda_n}{D-2}}\,, \qquad b' = \sqrt{\frac{\Lambda_m}{D-2}}\,. \tag{4.5}$$

For the parameters appropriate for the results plotted in figures 3 and 4 we obtain $a' = 1/\sqrt{2}$ and $b = 1/2$. We see that this is in agreement with the numerical results. It should be noted that eq. (4.5) seems to imply that we wouldn't see this behaviour for negatively curved submanifolds. This is indeed the case, for such metrics, and those where some of the submanifolds have zero curvature, we find only compact non-integrable configurations.

The low energy effective actions of the superstring in the Einstein frame are special cases of this type of system, if we keep only the kinetic terms for the form fields [29]. We have not been able to find any integrable singular instantons of the type described here at all for these systems. Naively, we might conclude that we can not use the instanton approximation to the Hartle Hawking wavefunction in string theory to verify the existence of hidden extra dimensions. However, there are a number of possible resolutions that have not yet been explored. These include the need to include corrections to the low energy effective action as we approach the singular pole. Unless these corrections stop the curvature from reaching the string scale we would have to include an infinite number of them — our effective field theory would break down altogether.

5. Conclusions

We have introduced antisymmetric tensor field degrees of freedom into the metric considered in [7]. In the case of a simple $F^2\sqrt{g}$ lagrangian term for these fields we find that they vanish in the dominant instanton. We have also studied the case where

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there is some kind of dilatonic coupling between the scalar field and the forms. In the case where the scalar fields potential is completely generated by its coupling to the forms we find no finite action singular instantons at all. Special cases of such systems include the naive string low energy effective actions. The main conclusion to be drawn from our calculations is that a naive application of the Hartle Hawking proposal to these higher-dimensional theories does not appear to lead to a universe with compactified extra dimensions as we would like. The dominant instanton in the path integral seems to have such a high degree of symmetry that it continues to a universe where none of the extra dimensions are compactified. It would be interesting to investigate the impact higher order corrections would have when applying the no boundary proposal in higher dimensions.

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